

# Bi-conformal symmetry and static Green functions in the higher-dimensional Reissner-Nordström spacetimes

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We study a static scalar massless field created by a source located near an electrically charged higher dimensional spherically symmetric black hole. We demonstrated that there exist bi-conformal transformations relating static field solutions in the metric with different parameters of the mass  $M$  and charge  $Q$ . Using this symmetry we obtain the static scalar Green function in the higher dimensional Reissner-Nordström spacetimes.

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## I. INTRODUCTION

In this paper we continue studying minimally coupled massless scalar fields created by static sources placed in the vicinity of a higher dimensional static black holes. For this purpose we use the method of bi-conformal transformations, which was developed and applied to the case of the Schwarzschild-Tangherlini metrics in our previous paper [1]. This method is based on the following observations.

A scalar massless field  $\Phi$  in a  $D$ -dimensional spacetime with metric  $g_{\mu\nu}$  ( $\mu, \nu = 0, \dots, D-1$ ) obeys the equation

$$\square\Phi = -4\pi J. \quad (1.1)$$

Let us consider a static scalar field  $\Phi(X)$  created by a source  $J(X)$  in the static spacetime with the metric

$$ds^2 = -\alpha^2 dt^2 + g_{ab} dx^a dx^b, \quad (1.2)$$

$$X = (t, x^a), \quad \alpha = \alpha(x), \quad g_{ab} = g_{ab}(x).$$

Then the equation (1.1) is reduced and takes the form

$$\hat{F}\Phi = -4\pi J, \quad \hat{F} = \frac{1}{\alpha\sqrt{g}}\partial_a(\alpha\sqrt{g}g^{ab}\partial_b). \quad (1.3)$$

Here  $g = \det(g_{ab})$ . The red-shift factor  $\alpha$  is connected with the norm of the static Killing vector  $\xi$  as follows:  $\alpha = \sqrt{-\xi^2} = \sqrt{-g_{tt}}$ . The equation (1.3) is invariant under the following *bi-conformal* transformations

$$\Phi = \bar{\Phi}, \quad g_{ab} = \Omega^2 \bar{g}_{ab}, \quad \alpha = \Omega^{-n} \bar{\alpha}, \quad J = \Omega^2 \bar{J}, \quad (1.4)$$

where  $n \equiv D-3$  and  $\Omega$  is an arbitrary function of spatial coordinates  $x^a$ .

This transformation consists of a bi-conformal map [2, 3] of the original background  $D$ -dimensional metric  $g_{\mu\nu}$

$$\Psi_\Omega : g \rightarrow \bar{g}, \quad (1.5)$$

accompanied by a properly chosen rescaling of the charge density  $J$ . If one starts with a solution of the Einstein equations, a new metric, obtained as a result of this transformation, is not necessarily a solution of the Einstein

equations with a physically meaningful stress-energy tensor. However, it may happen that for a specially chosen transformation this new metric has enhanced symmetries.

An interesting example is a Majumdar-Papapetrou metric, describing the gravitational field of a set of higher dimensional extremely charged black holes in equilibrium. Under properly chosen bi-conformal map this metric reduces to the higher dimensional Minkowski metric. This allows one to solve the static scalar field equation in the Majumdar-Papapetrou exactly (see [4]).

In the paper [1] we demonstrated that the method of bi-conformal transformations can be used for solving static equations in spacetimes of static spherically symmetric black holes. The enhanced symmetry of the bi-conformal metric  $\bar{g}$  was used in that paper to obtain the static Green functions for the equation (1.3) in a higher dimensional Schwarzschild-Tangherlini spacetime. In this paper we demonstrate how this method works for the case a charged higher dimensional black hole.

There are many possible applications of the proposed result. One of them is an old problem of finding a self-energy and a self-force of charged particles near black holes [5–8]. In four dimension the closed form of the exact solution for the field of a point charges in the black hole geometry was obtained earlier [7–12].

The recent interest to the problem of a self-force is stimulated by a study of the back-reaction of the field on the particle moving near black holes [13] in connection with the gravitational wave emission by such particles. More recently, several publications discussed higher-dimensional aspects of this problem (see, e.g., [14, 15]). This study was stimulated by general interest to spacetimes and brane models with large extra dimensions.

The paper is organized as follows. In Section II we discuss bi-conformal transformations of higher dimensional spherically symmetric metrics and demonstrate that the Reissner-Nordström metrics are bi-conformally related to the higher dimensional Bertotti-Robinson metric. The latter is a product of 2D anti-de Sitter space and a sphere. Using this result we construct a bi-conformal map of the Reissner-Nordström metrics with different parameters of mass and charge. In Section III we obtain useful rep-

representations for static Green functions in a spacetime of static spherically symmetric higher dimensional charged black holes. Section IV contains example of calculations of the static Green functions for 4,5 and 6 dimensional black holes. Section V contains discussion of the obtained results and their possible generalizations.

## II. BI-CONFORMAL MAP AND SYMMETRY ENHANCEMENT OF STATIC SPHERICALLY SYMMETRIC SPACETIMES

### A. Symmetry enhancement condition

Let us consider an application of the method of the bi-conformal maps to the case of a general static spherically symmetric  $D$ -dimensional metric. The corresponding metric is

$$ds^2 = -f(r) dt^2 + w^{-1}(r) dr^2 + r^2 d\omega_{n+1}^2, \quad (2.1)$$

where  $n = D - 3$  and  $d\omega_{n+1}^2$  is the line element on a  $(n + 1)$ -dimensional unit sphere

$$d\omega_{n+1}^2 = d\theta_n^2 + \sin^2 \theta_n d\omega_n^2, \quad d\omega_0^2 = d\phi^2. \quad (2.2)$$

We denote  $\theta_0 \equiv \phi \in [0, 2\pi]$ . All other coordinates  $\theta_{i>0} \in [0, \pi]$ . This metric is invariant under time translations and spatial rotations. Since  $ds$ ,  $t$  and  $r$  have the same dimensionality of the length, the metric (2.1) can be presented in the form  $ds^2 = a^2 dS^2$ , where the dimensionless metric  $dS^2$  is obtained from (2.1) by substituting  $t \rightarrow t/a$  and  $r \rightarrow r/a$ , where  $a$  is an arbitrary constant parameter with the dimensionality of length.

Let us apply a bi-conformal transformation (1.4) to this metric with  $\Omega = r/a$ . This choice guarantees that  $\Omega$  is dimensionless. After this bi-conformal transformation one has

$$\begin{aligned} d\bar{s}^2 &= dh^2 + a^2 d\omega_{n+1}^2, \\ dh^2 &= -\left(\frac{r}{a}\right)^{2n} f(r) dt^2 + \frac{a^2}{r^2 w(r)} dr^2. \end{aligned} \quad (2.3)$$

The scalar curvature of the two-dimensional metric  $dh^2$  is

$$\begin{aligned} R &= -\frac{1}{2a^2 f^2} \{r f w' (2n f + r f') \\ &\quad + w [2r^2 f'' - r^2 (f')^2 + 2r(2n + 1) f f' + 4n^2 f^2]\}. \end{aligned} \quad (2.4)$$

Here and later  $(\dots)' = d(\dots)/dr$ . The metric  $dh^2$  possesses an enhanced symmetry if its 2D curvature  $R$  is constant. We denote its value by

$$R = -\frac{2}{b^2}, \quad (2.5)$$

where  $b$  is a constant of the dimensionality of the length. The equations (2.4) and (2.5) can be solved to determine

the function  $w$ . The result is

$$w = \left( \frac{a^2}{n^2 b^2} + \frac{C}{r^{2n} f} \right) \left( 1 + \frac{r f'}{2n f} \right)^{-2}. \quad (2.6)$$

Here  $C$  is an integration constant. Let us suppose that function  $f$  has the following asymptotic at infinity

$$f = f_0 + f_1 r^{-\gamma} + \dots, \quad \gamma \geq 1. \quad (2.7)$$

Then (2.6) shows that asymptotic value of  $w$  at the infinity is  $a^2/(n^2 b^2)$ . The spacetime does not have a solid angle deficit and is asymptotically flat only if

$$\frac{a}{nb} = 1. \quad (2.8)$$

In what follows we always assume that this conditions is satisfied.

By using the relation (2.6) one finds such functions  $\{f(r), w(r)\}$ , for which the bi-conformal transformation of the metric (2.1) has an enhanced symmetry. The corresponding metric

$$d\bar{s}^2 = dh^2 + a^2 d\omega_{n+1}^2 \quad (2.9)$$

is a direct sum of the two dimensional anti-de Sitter metric  $dh^2$  and the metric  $a^2 d\omega_{n+1}^2$  on  $(n + 1)$ -dimensional sphere. The ratio of the curvature radii for these two metrics is fixed by the condition (2.8). This metric describes a particular Bertotti-Robinson spacetime and can be written in the following canonical form

$$d\bar{s}^2 = a^2 \left[ \frac{1}{n^2} \left( -(\rho^2 - 1) d\bar{\sigma}^2 + \frac{1}{\rho^2 - 1} d\rho^2 \right) + d\omega_{n+1}^2 \right]. \quad (2.10)$$

Let us emphasize that the parameter  $a$  has dimensionality of the length and it is arbitrary.

### B. Bi-conformal map of Reissner-Nordström metric to the Bertotti-Robinson space

Let us consider a special case of the metric (2.1) with an extra condition

$$w = f. \quad (2.11)$$

For this choice the relation (2.6) becomes an equation which allows one to obtain the function  $f$ . The ordinary differential equation (2.6) with  $w = f$  is of the first order. Hence its solution besides the constant  $C$  contains another arbitrary integration constants  $C_1$ . It is possible to show that one can choose these constants so that the solution takes the form

$$f = 1 - \frac{2M}{r^n} + \frac{Q^2}{r^{2n}}. \quad (2.12)$$

For real positive  $M$  and real  $Q$ , which satisfies the condition  $|Q| \leq M$ , the metric (2.1) with (2.11) and (2.12) is

the metric of a higher dimensional spherically symmetric electrically charged black hole with  $M$  and  $Q$  being its mass and charge, respectively.

In order to rewrite the metric  $ds^2$ , obtained as a result of the bi-conformal map (2.3), in the standard (canonical) form (2.10) it is sufficient to make the following coordinate transformations

$$r^n = M + \mu\rho, \quad t = \frac{a^{n+1}}{n\mu} \bar{\sigma}, \quad \mu = \sqrt{M^2 - Q^2}. \quad (2.13)$$

We denote this bi-conformal map as follows

$$\Psi_\Omega : g_{M,Q} \rightarrow \bar{g}_{BR}, \quad \Omega = r/a. \quad (2.14)$$

### C. Bi-conformal transformations within the Reissner-Nordström family of solutions

The method of bi-conformal maps was used in the paper [1] to obtain static Green functions in the background of the higher dimensional Schwarzschild-Tangherlini spacetimes. For this purpose, one uses at first the enhanced symmetry of a related Bertotti-Robinson space to find the  $D$ -dimensional Green function in this space, and after this one obtains the static Green function by means of the dimensional reduction. One can apply the same method for finding static Green functions in the Reissner-Nordström geometry. However, there exist another much simpler way. One can generate the corresponding static Green function in the spacetime of charged black holes by using the already known Green function for the Schwarzschild-Tangherlini spacetime.

For this purpose let us notice that the canonical form (2.10) is *universal* in the following sense: It is the same for any Reissner-Nordström metric and it does not depend on its parameters  $M$  and  $Q$ . This observation opens an interesting possibility to relate metrics with different parameters. Let us introduce new coordinates  $\hat{t}$  and  $\hat{r}$

$$\hat{r}^n = \hat{M} + \hat{\mu}\rho, \quad \hat{t} = \frac{a^{n+1}}{n\hat{\mu}} \bar{\sigma}, \quad \hat{\mu} = \sqrt{\hat{M}^2 - \hat{Q}^2}, \quad (2.15)$$

and denote

$$\hat{\Omega} = \hat{r}/a. \quad (2.16)$$

Then one has the following bi-conformal map of the Reissner-Nordström metric with parameters  $\hat{M}$  and  $\hat{Q}$  to the canonical Bertotti-Robinson metric

$$\Psi_{\hat{\Omega}} : g_{\hat{M},\hat{Q}} \rightarrow \bar{g}_{BR}. \quad (2.17)$$

Combining the direct bi-conformal map (2.14) with the bi-conformal map, inverse to (2.17), one obtain a bi-conformal map

$$\Psi = \Psi_{\hat{\Omega}}^{-1} \circ \Psi_\Omega : g_{M,Q} \rightarrow g_{\hat{M},\hat{Q}}. \quad (2.18)$$

This bi-conformal map is a transformation of the original Reissner-Nordström metric with parameters  $M$  and  $Q$  to

a similar metric with different parameters  $\hat{M}$  and  $\hat{Q}$ . The static equation (1.3) is invariant under such a transformation provided one in addition properly transforms the source term  $J \rightarrow \hat{J}$ .

In other words the solutions for the static field  $\Phi$  in the original background space are simply related to solutions in a spacetime with modified parameters of the mass and the charge. In particular, if one knows the static Green function in the spacetime of uncharged black hole, one can obtain the static Green function for the charged black hole by using the above described transformations. In the next section we demonstrate how this method works in more detail.

## III. STATIC GREEN FUNCTIONS

### A. Bi-conformal map of static Green functions

Following the paper [1] we define a static Green function  $G(x, x')$  as follows

$$G(x, x') = \int_{-\infty}^{\infty} dt \mathbb{G}_{\text{Ret}}(t, x; 0, x'). \quad (3.1)$$

Here  $\mathbb{G}_{\text{Ret}}(t, x; 0, x')$  is a retarded Green function in the  $D$ -dimensional spacetime. This static Green function satisfies the equation

$$\hat{F} G(x, x') = -\frac{\delta(x - x')}{\alpha\sqrt{g}}. \quad (3.2)$$

In what follows, we assume that this Green function is decreasing when one of its parameters  $x$  tends to infinity and remains regular at the horizon (for more details see [1]).

The static Green function is simply related to the expression for a scalar field created by a point charge. The current of a static point charge  $q$  positioned at the point  $y$  reads

$$J(x) = q \frac{\delta(x - y)}{\sqrt{g}}. \quad (3.3)$$

In this case the scalar field at the point  $x$  takes the form[20]

$$\Phi(x) = 4\pi q \alpha(y) G(x, y). \quad (3.4)$$

The field of a distributed source  $q(y)$  can be easily obtained by integration over  $y$  of the right-hand side of this relation.

It is convenient to introduce a new radial variable  $\rho$  related to the radial coordinate  $r$  as follows (2.13)

$$\rho = \frac{r^n - M}{\mu}. \quad (3.5)$$

The Reissner-Nordström metric takes the form

$$ds^2 = -\frac{\mu^2(\rho^2 - 1)}{(M + \mu\rho)^2} dt^2 + (M + \mu\rho)^{2/n} \left[ \frac{1}{n^2(\rho^2 - 1)} d\rho^2 + d\omega_{n+1}^2 \right]. \quad (3.6)$$

The horizon corresponds to  $\rho = 1$  and the gravitational radius  $r_g$  is given by the expression  $r_g^n = M + \mu$ . The surface gravity at the horizon is

$$\kappa = \frac{n\mu}{r_g^{n+1}}. \quad (3.7)$$

In these coordinates the equation for the static Green function takes the form

$$\begin{aligned} [n^2(\rho^2 - 1) \partial_\rho^2 + 2n^2 \rho \partial_\rho + \Delta_\omega^{n+1}] G(x, x') \\ = -\frac{n}{\mu} \delta(\rho - \rho') \delta(\omega, \omega'). \end{aligned} \quad (3.8)$$

Here  $\Delta_\omega^{n+1}$  and  $\delta(\omega, \omega')$  are the Laplace operator and a covariant delta-function on the unit  $(n+1)$ -dimensional sphere, respectively,

$$\begin{aligned} \Delta_\omega^{n+1} &= \partial_{\theta_n}^2 + n \frac{\cos \theta_n}{\sin \theta_n} \partial_{\theta_n} + \frac{1}{\sin^2 \theta_n} \Delta_\omega^n, \\ \Delta_\omega^1 &= \partial_\phi^2, \\ \delta^{n+1}(\omega, \omega') &= \frac{\delta(\theta_n - \theta'_n)}{\sin^n \theta_n} \delta^n(\omega, \omega'), \\ \delta^1(\omega, \omega') &= \delta(\phi - \phi'). \end{aligned} \quad (3.9)$$

Because of the spherical symmetry of background geometry, the resulting static Green functions are the functions of radial coordinates of the observer  $\rho$ , the source  $\rho'$ , as well as the angular distance  $\gamma \equiv \gamma_{n+1}$  between the source and the observational point.

$$G(x, x') = G(\rho, \rho'; \gamma). \quad (3.10)$$

The angular distance on the  $(n+1)$  dimensional sphere can be written explicitly in terms of the angular coordinates (2.2)

$$\begin{aligned} \cos \gamma_{n+1} &= \cos \theta_n \cos \theta'_n + \sin \theta_n \sin \theta'_n \cos \gamma_n, \\ \gamma_0 &= \phi - \phi'. \end{aligned} \quad (3.11)$$

The canonical Bertotti-Robinson spacetime (2.10) is homogeneous. In the paper [1] we have used the knowledge of heat kernels on homogeneous spaces to derive the static Green functions. Now, using the bi-conformal symmetry of the static operator (1.3), we can use these results to derive the static Green functions in the Reissner-Nordström spacetime with arbitrary parameters of the mass  $M$  and the charge  $Q$ .

One can see that the bi-conformal transformation (1.4) with

$$\Omega = \frac{r}{a} = \frac{(M + \mu\rho)^{1/n}}{a}, \quad (3.12)$$

leads to the Bertotti-Robinson canonical metric (2.10), if  $\bar{\sigma}$  is identified with the rescaled Reissner-Nordström time coordinate  $t$

$$\bar{\sigma} = \bar{\kappa} t, \quad \bar{\kappa} = \frac{n\mu}{a^{n+1}} = \left( \frac{r_g}{a} \right)^{n+1} \kappa. \quad (3.13)$$

Here  $\kappa$  is given by (3.7) and  $\bar{\kappa}$  is the surface gravity of the horizon  $\rho = 1$  in the Bertotti-Robinson spacetime, normalized according to the Killing vector  $\xi^\mu = \delta_t^\mu$

$$\bar{\kappa}^2 = -\frac{1}{2} \bar{\xi}^{\alpha;\beta} \bar{\xi}_{\alpha;\beta} \Big|_{\rho=1}. \quad (3.14)$$

One can define the static Green function in the canonical metric (2.10) as the integral over the dimensionless time coordinate  $\bar{\sigma}$

$$\bar{G}(x, x') = \int_{-\infty}^{\infty} d\bar{\sigma} \bar{\mathbb{G}}_{\text{Ret}}(\bar{\sigma}, x; 0, x'), \quad (3.15)$$

where  $\bar{\mathbb{G}}_{\text{Ret}}(\bar{\sigma}, x; \bar{\sigma}', x')$  is a retarded Green function in the canonical Bertotti-Robinson spacetime (2.10). It satisfies the the equation

$$\begin{aligned} [n^2(\rho^2 - 1) \partial_\rho^2 + 2n^2 \rho \partial_\rho + \Delta_\omega^{n+1}] \bar{G}(x, x') \\ = -\frac{n^2}{a^{n+1}} \delta(\rho - \rho') \delta(\omega, \omega'). \end{aligned} \quad (3.16)$$

The left-hand side of this equation coincides with that of (3.8). The right-hand sides of these equations differ only by a constant factor related to the rescaling of the time coordinate (3.13). Thus, the static Green functions in these spaces also differ only by a constant factor

$$G(\rho, \rho'; \gamma) = \frac{1}{\bar{\kappa}} \bar{G}(\rho, \rho'; \gamma). \quad (3.17)$$

To construct the bi-conformal map (2.18) relating Reissner-Nordström with different parameters  $M$  and  $Q$  one proceeds as follows. Let us define two radial coordinates  $r$  and  $\hat{r}$  by the relation

$$\frac{r^n - M}{\mu} = \frac{\hat{r}^n - \hat{M}}{\hat{\mu}} \equiv \rho. \quad (3.18)$$

Here

$$\mu = \sqrt{M^2 - Q^2}, \quad \hat{\mu} = \sqrt{\hat{M}^2 - \hat{Q}^2}. \quad (3.19)$$

This allows one to express the new coordinate  $\hat{r}$  in terms of the original radial coordinate  $r$ . The time coordinates are related as follows

$$\hat{t} = \frac{\mu}{\hat{\mu}} t. \quad (3.20)$$

The the bi-conformal transformation with

$$\Omega = \left[ \frac{M + \mu\rho}{\hat{M} + \hat{\mu}\rho} \right]^{1/n} \quad (3.21)$$

relates two arbitrary Reissner-Nordström metrics (3.6) characterized by the parameters  $M, Q$  and  $\hat{M}, \hat{Q}$ , correspondingly.

Because of the time rescaling the relation between the static Green functions in these Reissner-Nordström spacetimes becomes

$$\mu G(r, r'; \gamma) = \hat{\mu} \hat{G}(\hat{r}, \hat{r}'; \gamma). \quad (3.22)$$

Note that though the static Green function depends on the time rescaling, this dependence is dropped out of the expression for the scalar field  $\Phi$ . The resulting  $\Phi$  is invariant with respect to the time rescaling. One can say that the static scalar potentials  $\Phi$  for all Reissner-Nordström geometries are given by the same function of  $\rho$ . In terms of the radial coordinates  $r$  and  $\hat{r}$  they are related by the coordinate transformation (3.18). Therefore, as soon as we know the static scalar Green function for a particular choice of the charge of a black hole, for example, for a neutral one, the identity (3.22) makes it possible to generate the solution for the scalar field near the Reissner-Nordström black hole with an arbitrary mass and charge.

Since the cases of even and odd-dimensional spacetimes differ, so that we shall treat them separately.

### B. Even-dimensions

In even dimensions the exact static Green function can be represented in the form of the integral

$$G(x, x') = \frac{1}{n\mu} \frac{1}{2(2\pi)^{\frac{n+3}{2}}} \left( \frac{\partial}{\partial \cos \gamma} \right)^{(n+1)/2} \int_0^{2\pi} d\sigma A_n. \quad (3.23)$$

Here  $n = D - 3$  and

$$\cosh(\chi) = \rho\rho' - \sqrt{\rho^2 - 1}\sqrt{\rho'^2 - 1} \cos \sigma. \quad (3.24)$$

When  $n \geq 2$ , the functions  $A_n(\sigma, \rho, \rho'; \gamma)$  are given by the integral

$$A_n = \int_{\chi}^{\infty} dy \frac{1}{\sqrt{\cosh(y) - \cosh(\chi)}} \frac{\sinh\left(\frac{y}{n}\right)}{\sqrt{\cosh\left(\frac{y}{n}\right) - \cos(\gamma)}}. \quad (3.25)$$

At large  $y$  the integrand in (3.25) behaves like  $\exp[-y(n-1)/(2n)]$ . Therefore, (3.25) is convergent for any  $n \geq 2$ . In the case of the four-dimensional spacetime ( $n = 1$ ) the integrand has to be modified to guarantee convergence of the integral. For example, one can subtract the asymptotic of the integrand, which does not depend on  $\gamma$ . Since (3.23) contains the derivative of  $A_n$  over  $\gamma$ , the resulting Green function does not depend on the particular form of the subtracted  $\gamma$ -independent asymptotic. Thus, for

$n = 1$  one can choose

$$A_1 = \int_{\chi}^{\infty} dy \frac{\sinh(y)}{\sqrt{\cosh(y) - \cosh(\chi)}} \times \left[ \frac{1}{\sqrt{\cosh(y) - \cos(\gamma)}} - \frac{1}{\sqrt{\cosh(y) + 1}} \right]. \quad (3.26)$$

Substitution

$$\rho = \frac{r^n - M}{\mu} = \frac{r^n - M}{\sqrt{M^2 - Q^2}} \quad (3.27)$$

into (3.23) gives the static Green function of a scalar charge near the Reissner-Nordström black hole (3.6) in terms of the radial coordinate  $r$ .

### C. Odd-dimensions

In odd-dimensional spacetimes we have

$$G(x, x') = \frac{1}{n\mu} \frac{1}{\sqrt{2}(2\pi)^{\frac{n+4}{2}}} \left( \frac{\partial}{\partial \cos \gamma} \right)^{n/2} \int_0^{2\pi} d\sigma B_n, \quad (3.28)$$

where  $n = D - 3$  and  $\chi$  is given by (3.24) and

$$B_n = \int_{\chi}^{\infty} dy \frac{1}{\sqrt{\cosh y - \cosh \chi}} \frac{\sinh\left(\frac{y}{n}\right)}{\cosh\left(\frac{y}{n}\right) - \cos \gamma}. \quad (3.29)$$

Similarly to the even dimensions, the Green function in the radial coordinates  $r$  can be obtained after the substitution (3.27).

## IV. CLOSED FORM OF THE GREEN FUNCTION: EXAMPLES

### A. Four dimensions. $D = 4$

In four dimensions ( $n = 1$ ) the integral (3.26) can be done and one obtains

$$A_1 = \ln \left( \frac{\cosh(\chi) + 1}{\cosh(\chi) - \cos(\gamma)} \right). \quad (4.1)$$

The integral over  $\sigma$  can be taken explicitly and we obtain the closed form for the static Green function

$$G(x, x') = \frac{1}{4\pi\mu} \frac{1}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos \gamma - 1 + \cos^2 \gamma}}, \quad (4.2)$$

$$\rho = \frac{r - M}{\mu} = \frac{r - M}{\sqrt{M^2 - Q^2}}.$$

When written in terms of the radial coordinate  $r$  it reads

$$G(x, x') = \frac{1}{4\pi\mathcal{R}}, \quad (4.3)$$

where

$$\begin{aligned} \mathcal{R}^2 &= (r - M)^2 + (r' - M)^2 \\ &\quad - 2(r - M)(r' - M) \cos \gamma - (M^2 - Q^2) \sin^2 \gamma. \end{aligned} \quad (4.4)$$

This formula exactly reproduces the closed form of the well known result for the scalar Green function in four-dimensional Schwarzschild geometry [8, 11, 16]. It is easy to check that using the bi-conformal symmetry (3.22) this solution could be generated from that of the Schwarzschild case ( $Q = 0$ ).

In the limit of the extremally charged black hole  $Q = M$  the obtained solution (4.2) reproduces the result [4] for the four-dimensional Majumdar-Papapetrou geometry.

### B. Five dimensions. $D = 5$

The other case, when there exists a closed form for the static Green function is five-dimensional ( $n = 2$ ) Reissner-Nordström black hole. One can generate this solution using the bi-conformal symmetry (3.22) from that of the Tangherlini black hole [1], or, equivalently, just make the substitution (3.27) in the expression for the five-dimensional Green function (see eq.(6.14) of [1]).

$$\begin{aligned} G(x, x') &= \frac{1}{8\pi^2\mu} \frac{1}{(\rho^2 - 1)^{1/4}(\rho'^2 - 1)^{1/4}} \\ &\quad \times \frac{\partial}{\partial \cos \gamma} \{ \varkappa [\mathbf{F}(\psi, \varkappa) + \mathbf{K}(\varkappa)] \}, \end{aligned} \quad (4.5)$$

where  $\mathbf{F}$  and  $\mathbf{K}$  are the elliptic functions

$$\rho = \frac{r^2 - M}{\mu}, \quad (4.6)$$

and

$$\begin{aligned} \sin \psi &= \cos \gamma \frac{\sqrt{2}}{\sqrt{\rho\rho' - \sqrt{\rho^2 - 1}\sqrt{\rho'^2 - 1} + 1}}, \\ \varkappa &= \frac{\sqrt{2}(\rho^2 - 1)^{1/4}(\rho'^2 - 1)^{1/4}}{\sqrt{\rho\rho' + \sqrt{\rho^2 - 1}\sqrt{\rho'^2 - 1} + 1 - 2\cos^2 \gamma}}. \end{aligned} \quad (4.7)$$

To the best of our knowledge, this closed form for the static Green function in five-dimensional Reissner-Nordström black hole is new.

In the limit of the extremally charged black hole, when  $Q = M$ , the expression (4.5) leads to

$$G(x, x') = \frac{1}{4\pi^2\mathcal{R}^2}, \quad (4.8)$$

where

$$\begin{aligned} \mathcal{R}^2 &= (r^2 - M) + (r'^2 - M) \\ &\quad - 2\cos \gamma \sqrt{r^2 - M} \sqrt{r'^2 - M}. \end{aligned}$$

It exactly reproduces the result [4] for the five-dimensional Majumdar-Papapetrou geometry in the case of a single extremal black hole of the mass  $M$ .

### C. Six dimensions. $D = 6$

Application of the (3.23) to six-dimensional ( $n = 3$ ) Reissner-Nordström black hole leads to

$$\begin{aligned} A_3 &= \int_{\chi}^{\infty} dy \frac{1}{(\cosh y - \cosh \chi)^{1/2}} \frac{\sinh(\frac{y}{3})}{\sqrt{\cosh(\frac{y}{3}) - \cos(\gamma)}} \\ &= 3 \int_{\cosh(\chi/3)}^{\infty} dz \frac{1}{\sqrt{4z^3 - 3z - \cosh \chi}} \frac{1}{\sqrt{z - \cos \gamma}}. \end{aligned} \quad (4.9)$$

This integral can be expressed in terms of the elliptic function  $\mathbf{F}$

$$A_3 = \frac{6}{\sqrt{v(w - u)}} \mathbf{F} \left( \arcsin \sqrt{\frac{w - u}{w}}, \frac{w(v - u)}{v(w - u)} \right), \quad (4.10)$$

where

$$\begin{aligned} p &= \cosh(\chi/3), & w &= 2(p - \cos \gamma), \\ u &= 3p - i\sqrt{3p^2 - 3}, & v &= 3p + i\sqrt{3p^2 - 3}. \end{aligned} \quad (4.11)$$

Note that  $A_3$  is real in spite of the complexity of the functions  $u$  and  $v$ . Thus the static Green function in the six-dimensional Schwarzschild-Tangherlini spacetime is given by the integral

$$G(x, x') = \frac{1}{48\pi^3\mu} \left( \frac{\partial}{\partial \cos \gamma} \right)^2 \int_0^{2\pi} d\sigma A_3. \quad (4.12)$$

It is problematic to obtain an answer for the Green functions in a closed form for  $D \geq 6$ . However, a rather simple integral representation is possible in all higher dimensions. For some applications, like computing of the self-force and self-energy of scalar charges this integral representations is sufficient to obtain the final results in a closed form.

## V. DISCUSSION

In this paper we demonstrated that there exist bi-conformal transformations relating static solutions of the minimally coupled massless field equation in the Reissner-Nordström spacetimes with different values of the parameters of the mass  $M$  and the charge  $Q$ . We used this symmetry to generate expressions for the static Green functions in such space starting from similar Green functions for the neutral (uncharged) higher dimensional black holes, that have been obtained earlier [1]. To check the obtained results, we considered limit of higher dimensional extreme black holes with  $|Q| = M$ . This is a special case of the Majumdar-Papapetrou metrics related by means of a bi-conformal map to the flat spacetime. It is possible to show that the obtained static Green functions in a generic Reissner-Nordström spacetime obey a correct flat spacetime limit.

Natural applications of the results obtained in our earlier publication [1] and in this paper is study of the problem of the self-energy and self force of point scalar charged in the background of higher dimensional static black holes. Especially interesting is the origin of near horizon logarithmic terms in these expressions in odd dimensional black holes [14, 15] and the relation of these terms with the bi-conformal anomalies (see discussion in [17–19]). It is interesting also to test the method of the bi-conformal transformations in application to electric fields of static sources in the static black hole backgrounds. Another interesting question is: Is it possible to generalize the method of bi-conformal to the case of fields from sta-

tionary sources in a spacetime of rotating black holes. We are going to address these questions in our further work.

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  - [20] Note that rescaling of the time variable by a constant factor  $t = c\tilde{t}$  leads to  $\tilde{\alpha} = c\alpha$ ,  $\tilde{J} = J$ ,  $\tilde{G}(x, x') = c^{-1}G(x, x')$ ,  $\tilde{\Phi} = \Phi$ . Thus the scalar field  $\Phi$  created by the static source is invariant under this constant rescaling of the time coordinate.